Mixed Covering Arrays on 3-Uniform Hypergraphs

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Abstract

Covering arrays are combinatorial objects that have been successfully applied in the design of test suites for testing systems such as software, circuits and networks, where failures can be caused by the interaction between their parameters. In this paper, we perform a new generalization of covering arrays called covering arrays on 3-uniform hypergraphs. Let n, k be positive integers with $k \geq 3$. Three vectors $x \in \mathbb{Z}_{g_1}^n$, $y \in \mathbb{Z}_{g_2}^n$, $z \in \mathbb{Z}_{g_3}^n$ are 3-qualitatively independent if for any triplet $(a, b, c) \in \mathbb{Z}_{g_1} \times \mathbb{Z}_{g_2} \times \mathbb{Z}_{g_3}$, there exists an index $j \in \{1, 2, ..., n\}$ such that $(x(j), y(j), z(j)) = (a, b, c)$. Let H be a 3-uniform hypergraph with k vertices $v_1, v_2, \ldots, v_k$ with respective vertex weights $g_1, g_2, \ldots, g_k$. A mixed covering array on H, denoted by $3-CA(n, H, \prod_{i=1}^{k} g_i)$, is a $k \times n$ array such that row i corresponds to vertex v_i, entries in row i are from Z_{g_i}; and if $\{v_x, v_y, v_z\}$ is a hyperedge in H, then the rows x, y, z are 3-qualitatively independent. The parameter n is called the size of the array. Given a weighted 3-uniform hypergraph H, a mixed covering array on H with minimum size is called optimal. We outline necessary background in the theory of hypergraphs that is relevant to the study of covering arrays on hypergraphs. In this article, we introduce five basic hypergraph operations to construct optimal mixed covering arrays on hypergraphs. Using these operations, we provide constructions for optimal mixed covering arrays on α-acyclic 3-uniform hypergraphs, conformal 3-uniform hypertrees having a binary tree as host tree, and on some specific 3-uniform cycle hypergraphs.

Keywords: Covering arrays, host graph, conformal 3-uniform hypertrees, α-acyclic 3-uniform hypergraphs, 3-uniform cycles, software testing.

1 Introduction

Covering arrays have been extensively studied and have been the topic of interest of many researchers. These interesting mathematical structures are generalizations of well known orthogonal arrays [16]. A covering array of strength three, denoted by 3-$CA(n, k, g)$, is an $k \times n$ array C with entries from $\mathbb{Z}_g$ such that any three distinct rows of C are 3-qualitatively independent. The parameter n is called the size of the array. *One of the main problems on covering arrays is to construct a 3-$CA(n, k, g)$ for given parameters (k, g) so that the size n is as small as possible.* The covering array number 3-$CAN(k, g)$ is the smallest n for which a 3-$CA(n, k, g)$ exists, that is

$$3\text{-}CAN(k, g) = min_{n \in \mathbb{N}} \{ n \mid \exists\ 3\text{-}CA(n, k, g) \}.$$

A 3-$CA(n, k, g)$ of size $n =$ 3-$CAN(k, g)$ is called $optimal$. An example of a strength three covering array 3-$CA(10, 5, 2)$ is shown below [5]:

$$\begin{array}{cccccccccc}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0
\end{array}$$

There is a vast array of literature [14, 9, 10, 4, 5, 21] on covering arrays, and the problem of determining the minimum size of covering arrays has been studied under many guises over the past thirty years.

Covering arrays have applications in many areas. Covering arrays are particularly useful in the design of test suites [14, 7, 8, 19, 17, 18]. The testing application is based on the following translation. Consider a software system that has k parameters, each parameter can take g values. Exhaustive testing would require g^k test cases for detecting software failure, but if k or g are reasonably large, this may be infeasible. We wish to build a test suite that tests all 3-way interactions of parameters with the minimum number of test cases. Covering arrays of strength 3 provide compact test suites that guarantee 3-way coverage of parameters.

Several generalizations of covering arrays have been proposed in order to address different requirements of the testing application (see [9, 15]). *Mixed covering arrays* are a generalization of covering arrays that allows different values for different rows. This meets the requirement that different parameters in the system may take a different number of possible values. Constructions for mixed covering arrays are given

in [11, 22]. Another generalization of covering arrays are *mixed covering arrays on hypergraph*. In these arrays, only specified choices of distinct rows need to be qualitatively independent and these choices are recored in hypergraph. As mentioned in [20], this is useful in situations in which some combinations of parameters do not interact; in these cases, we do not insist that these interactions to be tested, which allows reductions in the number of required test cases. This has been applied in the context of software testing by observing that we only need to test interactions between parameters that jointly effect one of the output values [6]. Covering arrays on graphs were first studied by Serroussi and Bshouty [24], who showed that finding an optimal covering array on a graph is NP-hard for the binary case. Covering arrays on general alphabets have been systematically studied in Steven's thesis [25]. Meagher and Stevens [21], and Meagher, Moura, and Zekaoui [20] studied strength two (mixed) covering arrays on graphs in more details and gave many powerful results. Variable strength covering arrays have been introduced and systematically studied in Raaphorst's thesis [23].

In this paper, we extend the work done by Meagher, Moura, and Zekaoui [20] for mixed covering arrays on graph to mixed covering arrays on hypergarphs. The motivation for this generalisation is to improve applications of covering arrays to software, circuit and network systems. This extension also gives us new ways to study covering arrays construction. In Section 2, we outline necessary background in the theory of hypergraphs and mixed covering arrays that are relevant to the study of mixed covering arrays on hypergraphs. In Section 3, we present results related to balanced and pairwise balanced vectors which are required for basic hypergraph operations. In section 4, we introduce four basic hypergraph operations. Using these operations, we construct optimal mixed covering arrays on α-acyclic 3-uniform hypergraphs, conformal 3-uniform hypertrees having a binary tree as host tree, some specific 3-uniform cycles. In Section 5, we build optimal mixed covering arrays on 3-uniform cycles with exactly one vertex of degree one.

2 Mixed covering arrays and hypergraphs

A mixed covering array is a generalization of covering array that allows different alphabets in different rows.

Definition 1. *(Mixed Covering Array) Let $n, k, g_1, \ldots, g_k$ be positive integers. A mixed covering array of strength three, denoted by $3-CA(n, k, \prod_{i=1}^{k} g_i)$ is an $k \times n$ array C with entries from $\mathbb{Z}_{g_i}$ in row i, such that any three distinct rows of C are 3-qualitatively independent.*

The parameter n is called the size of the array. An obvious lower bound for the size of a covering array

is $g_i g_j g_k$ where g_i, g_j, g_k are the largest three alphabets, in order to guarantee that the corresponding three rows be 3-qualitatively independent.

Definition 2. *(Hypergraphs [2]) A hypergraph H is a pair $H = (V, E)$ where $V = \{v_1, v_2, \ldots, v_k\}$ is a set of elements called nodes or vertices, and $E = \{E_1, E_2, \ldots, E_m\}$ is a set of non-empty subsets of V, called hyperedges, such that*

$$E_i \neq \emptyset \quad (i = 1, 2, \ldots m)$$

$$\bigcup_{i=1}^{m} E_i = V.$$

A simple hypergraph is a hypergraph H such that

$$E_i \subset E_j \Rightarrow i = j.$$

If cardinality of every hyperedge of H is equal to r then H is called r-uniform hypergraph. A complete r-uniform hypergraph containing k vertices, denoted by K_k^r, is a hypergraph having every r-subset of set of vertices as hyperedge. For a set $J \subset \{1, 2, ..., m\}$, the partial hypergraph generated by J is the hypergraph $(V, \{E_i | i \in J\})$. For a set $A \subset V$, the subhypergraph H_A induced by A is defined as $H_A = (A, \{E_j \cap A \mid 1 \leq i \leq m, E_i \cap A \neq \emptyset\})$. The 2-section of a hypergraph H is the graph $[H]_2$ with the same vertices of the hypergraph, and edges between all pairs of vertices contained in the same hyperedge.

Definition 3. *(Conformal Hypergraph [2]) A hypergraph H is conformal if all the maximal cliques of the graph $[H]_2$ are hyperedges of H.*

Definition 4. *(Tripartite 3-uniform hypergraph [2]) A tripartite 3-uniform hypergraph is a 3-uniform hypergraph in which the set of vertices is $V_1 \cup V_2 \cup V_3$ and the hyperedges are the 3-tuples $\{v_1, v_2, v_3\}$ with $v_i \in V_i$ for $i = 1, 2, 3$.*

Definition 5. *[2] Let H be a hypergraph on V, and let $k \geq 2$ be an integer. A cycle of length k is a sequence $(v_1, E_1, v_2, E_2, ..., v_k, E_k, v_1)$ with:*

1. *$E_1, E_2, ..., E_k$ distinct hyperedges of H;*

2. *$v_1, v_2, ..., v_k$ distinct vertices of H;*

3. *$v_i, v_{i+1} \in E_i$ for $i = 1, 2, \ldots, k-1$;*

4. $v_k, v_1 \in E_k$.

Definition 6. *(Balanced Hypergraphs [2]) A hypergraph is said to be balanced if every odd cycle has a hyperedge containing three vertices of the cycle.*

Theorem 1. *[2] A hypergraph is balanced if and only if its induced subhypergraphs are 2-colourable.*

A *vertex-weighted hypergraph* is a hypergraph with a positive weight assigned to each vertex. We give here the definition of mixed covering array on hypergraph:

Definition 7. *Let H be a vertex-weighted hypergraph with k vertices and weights $g_1 \leq g_2 \leq \ldots \leq g_k$, and let n be a positive integer. A covering array on H, denoted by $CA(n, H, \prod_{i=1}^{k} g_i)$, is an $k \times n$ array with the following properties:*

1. *the entries in row i are from $\mathbb{Z}_{g_i}$;*
2. *row i corresponds to a vertex $v_i \in V(H)$ with weight g_i;*
3. *if $e = \{v_1, v_2, \ldots, v_t\} \in E(H)$, the rows correspond to vertices $v_1, v_2, \ldots, v_t$ are t-qualitatively independent.*

In this paper we concentrate on covering arrays on 3-uniform hypergraphs. Given a weighted 3-uniform hypergraph H with weights $g_1, g_2, \ldots, g_k$ a *strength-3 mixed covering array* on H is denoted by 3-$CA(n, H, \prod_{i=1}^{k} g_i)$; *the strength-3 mixed covering array number* on H, denoted by 3-$CAN(H, \prod_{i=1}^{k} g_i)$, is the minimum n for which there exists a 3-$CA(n, H, \prod_{i=1}^{k} g_i)$. A 3-$CA(n, H, \prod_{i=1}^{k} g_i)$ of size n = 3-$CAN(H, \prod_{i=1}^{k} g_i)$ is called $optimal$. A mixed covering array of strength three, denoted by 3-$CA(n, k, \prod_{i=1}^{k} g_i)$, is a 3-$CA(n, K_k^3, \prod_{i=1}^{k} g_i)$, where K_k^3 is the complete 3-uniform hypergraph on k vertices with weights g_i, for $1 \leq i \leq k$.

3 Balanced and Pairwise Balanced Vectors

In this section, we present several results related to balanced and pairwise balanced vectors which are required for basic hypergraph operations defined in the next section.

Definition 8. *A length-n vector with alphabet size g is balanced if each symbol occurs $\lfloor n/g \rfloor$ or $\lceil n/g \rceil$ times.*

Definition 9. *Two length-n vectors x_1 and x_2 with alphabet size g_1 and g_2 are pairwise balanced if both vectors are balanced and each pair of alphabets $(a, b) \in \mathbb{Z}_{g_1} \times \mathbb{Z}_{g_2}$ occurs $\lfloor n/g_1g_2 \rfloor$ or $\lceil n/g_1g_2 \rceil$ times in (x_1, x_2), so for $n \geq g_1g_2$ pairwise balanced vectors are always 2-qualitatively independent.*

Definition 10. *Let H be a vertex-weighted hypergraph. A balanced covering array on H is a covering array on H in which every row is balanced and the rows correspond to vertices in a hyperedge are pairwise balanced.*

Lemma 1. *Let $x_1 \in \mathbb{Z}_{g_1}^n$ and $x_2 \in \mathbb{Z}_{g_2}^n$ be two balanced vectors. Then for any positive integer h, there exists a balanced vector $y \in \mathbb{Z}_h^n$ such that x_1 and y are pairwise balanced and x_2 and y are pairwise balanced.*

Proof. Construct a bipartite multigraph G corresponds to x_1 and x_2 as follow: G has g_1 vertices in the first part $P \subseteq V(G)$ and g_2 vertices in the second part $Q \subseteq V(G)$. Let $P_a = \{i \mid x_1(i) = a\}$ for $a = 0, 1, \ldots, g_1 - 1$, be the vertices of P, while $Q_b = \{i \mid x_2(i) = b\}$ for $b = 0, 1, \ldots, g_2 - 1$, be the vertices of Q. We have that $\lfloor \frac{n}{g_1} \rfloor \leq |P_a| \leq \lceil \frac{n}{g_1} \rceil$ and $\lfloor \frac{n}{g_2} \rfloor \leq |Q_b| \leq \lceil \frac{n}{g_2} \rceil$, as x_1 and x_2 are balanced vectors. For each $i = 1, 2, \ldots, n$ there exists exactly one $P_a \in P$ with $i \in P_a$ and exactly one $Q_b \in Q$ with $i \in Q_b$. For each such i, add an edge between vertices corresponding to P_a and Q_b and label it i. Hence $d_G(P_a) = |P_a|$ and $d_G(Q_b) = |Q_b|$. If any vertex v of G has $d_G(v) > h$ then we split it into $\lfloor \frac{d_G(v)}{h} \rfloor$ vertices of degree h and, if necessary, one vertex of degree $d_G(v) - h\lfloor \frac{d_G(v)}{h} \rfloor$. Denote this resultant bipartite multigraph by H with maximum degree $\Delta(H) = h$. We know that a bipartite graph H with maximum degree h is the union of h matching. Thus $E(H)$ is union of h matchings F_0, F_1, …, F_{h-1}. Now identify those points of H which corresponds to the same point of G, then F_0, F_1, …, F_{h-1} are mapped onto certain edge disjoint spanning subgraphs F'_0, F'_1, …, F'_{h-1} of G. These h edge-disjoint spanning subgraphs F'_0, F'_1, …, F'_{h-1} of G form a partition of $E(G) = [1, n]$ which we use to build a balanced vector $y \in \mathbb{Z}_h^n$. Each edge disjoint spanning subgraph corresponds to a symbol in $\mathbb{Z}_h$ and each edge corresponds to an index from $[1, n]$. Suppose edge disjoint spanning subgraph F'_c corresponds to symbol $c \in \mathbb{Z}_h$. For each edge i in F'_c, define $y(i) = c$. Since F_i is a matching, there is atmost one F_i-edge incident with any of the $\lceil \frac{d_G(P_a)}{h} \rceil$ vertices of H corresponds to $P_a \in P$. Hence

$$d_{F'_i}(P_a) \leq \lceil \frac{d_G(P_a)}{h} \rceil.$$

On the other hand, there are $\lfloor \frac{d_G(P_a)}{h} \rfloor$ vertices of H corresponds to P_a which have degree h. There must be

an F_i-edge starting from each of these, whence

$$d_{F_i'}(P_a) \geq \lfloor \frac{d_G(P_a)}{h} \rfloor.$$

Thus we have $\lfloor \frac{n}{g_1 h} \rfloor \leq d_{F_i'}(P_a) \leq \lceil \frac{n}{g_1 h} \rceil$ for $i = 0, 1, \ldots, h-1$. This means that there exist $\lfloor \frac{n}{g_1 h} \rfloor$ or $\lceil \frac{n}{g_1 h} \rceil$ edges $i \in [1, n]$ such that $x_1(i) = a$ and $y(i) = c$, or in other words, each pair of symbols $(a, c) \in \mathbb{Z}_{g_1} \times \mathbb{Z}_h$ between x_1 and y appears either $\lfloor \frac{n}{g_1 h} \rfloor$ or $\lceil \frac{n}{g_1 h} \rceil$ times. So, x_1 and y are pairwise balanced vectors. Similarly, we can show that y and x_2 are pairwise balanced vectors. Next, we need to show that y is balanced. This corresponds to each spanning subgraph F_i' contains either $\lfloor \frac{n}{h} \rfloor$ or $\lceil \frac{n}{h} \rceil$ edges. In other words, this corresponds to each matching F_i contains either $\lfloor \frac{n}{h} \rfloor$ or $\lceil \frac{n}{h} \rceil$ edges. Suppose we have two matchings F_0 and F_1 that differ by size more than 1, say F_0 smaller and F_1 larger. Every component of the union of F_0 and F_1 could be an alternating even cycle or an alternating path. Note that it must contain a path, otherwise their sizes are equal. We can find a path component in the union graph that contains more edges from F_1 than F_0. Swap the F_1 edges with the F_0 edges in this path component. Then the resultant graph has F_0 increased in size by 1 edge, and F_1 decreased in size by 1 edge. Continue this process on F_0, F_1, …, F_{h-1} until the sizes are correct. □

The following corollary is an easy consequence of Lemma 1.

Corollary 1. *Let $x \in \mathbb{Z}_g^n$ be a balanced vector. Then for any positive integer h, there exists a balanced vector $y \in \mathbb{Z}_h^n$ such that x and y are pairwise balanced.*

Proof. This follows from Lemma 1. Set $x_1 = x$ and $x_2 = x$. □

Lemma 2. *Let $x_1 \in \mathbb{Z}_{g_1}^n$ and $x_2 \in \mathbb{Z}_{g_2}^n$ be two pairwise balanced vectors. Then for any h such that $g_1 g_2 h \leq n$, there exists a balanced vector $y \in \mathbb{Z}_h^n$ such that x_1, x_2 and y are 3-qualitatively independent and x_1 and y are pairwise balanced and x_2 and y are pairwise balanced.*

Proof. Construct a bipartite multigraph G corresponds to x_1 and x_2 as defined in the proof of Lemma 1. We have that the vectors x_1 and x_2 are pairwise balanced, that is, for each pair $(a, b) \in \mathbb{Z}_{g_1} \times \mathbb{Z}_{g_2}$, the number of edges between P_a and Q_b is $\lfloor \frac{n}{g_1 g_2} \rfloor$ or $\lceil \frac{n}{g_1 g_2} \rceil$. The problem is to find a balanced vector $y \in \mathbb{Z}_h^n$, such that x_1, x_2 and y are 3-qualitatively independent, x_1 and y are pairwise balanced, and x_2 and y are pairwise balanced. Assume without loss of generality that $g_1 \leq g_2$. We construct a bipartite multigraph H from G

as follow: We split each point $P_a \in P$ in G into $\lfloor \frac{d_G(P_a)}{h} \rfloor$ points of degree h and, if necessary, one point of degree $d_G(P_a) - h\lfloor \frac{d_G(P_a)}{h} \rfloor$ in H. Thus, using balancedness of x_1, we have that there are at least g_2 copies of P_a in H from the split operation. Label them P_{a0}, P_{a1}, ..., P_{a,g_2-1}, $P_{ag_2} \dots$ (g_2 onwards are extra). Similarly we split each point $Q_b \in Q$ into $\lfloor \frac{d_G(Q_b)}{h} \rfloor$ points of degree h and, if necessary, one point of degree $d_G(Q_b) - h\lfloor \frac{d_G(Q_b)}{h} \rfloor$ in H. Thus, using balancedness of x_2, we have that there are at least g_1 copies of Q_b in H from the split operation. Label them Q_{b0}, Q_{b1}, ..., Q_{b,g_1-1}, $Q_{bg_1} \dots$ (g_1 onwards are extra). For each pair of vertices P_a and Q_b, we have at least h edges between P_a and Q_b; consider only the first h edges from P_a to Q_b (ignore the rest for now). These h edges between P_a and Q_b in G become the h edges between P_{ab} and Q_{ba} in H. This results in a graph (possibly multigraph) where every vertex has maximum degree h. We add remaining edges arbitrarily to H amongst the remaining vertices (including the extra vertices) in any way, provided we maintain H as bipartite graph with maximum degree h and every vertex v of G is split into $\lfloor \frac{d_G(v)}{h} \rfloor$ points of degree h and, if necessary, one point of degree $d_G(v) - h\lfloor \frac{d_G(v)}{h} \rfloor$. We know that a bipartite graph with maximum degree h is the union of h matching. Thus $E(H)$ is union of h matchings F_0, $F_1, \dots, F_{h-1}$. Now identify those points of H which corresponds to the same point of G, then $F_0, F_1, \dots,$ F_{h-1} are mapped onto certain edge disjoint spanning subgraphs $F'_0, F'_1, \dots, F'_{h-1}$ of G. We claim each of the spanning subgraphs F'_i is a complete bipartite multigraph. For every $a \in \mathbb{Z}_{g_1}$, $b \in \mathbb{Z}_{g_2}$, there are h edges from P_{ab} to Q_{ba} in H, and they will all appear in different matchings $F_0, F_1, \dots, F_{h-1}$. This ensures that the spanning subgraphs contain at least one $P_a - Q_b$ edge for every $a \in \mathbb{Z}_{g_1}$, $b \in \mathbb{Z}_{g_2}$. This proves that each of the spanning subgraphs F'_i is a complete bipartite multigraph. These h edge-disjoint spanning subgraphs $F'_0, F'_1, \dots, F'_{h-1}$ of G form a partition of $E(G) = [1, n]$ which we use to build a balanced vector $y \in \mathbb{Z}_h^n$. Each edge disjoint spanning subgraph corresponds to a symbol in $\mathbb{Z}_h$ and each edge corresponds to an index from $[1, n]$. Suppose edge disjoint spanning subgraph F'_c corresponds to symbol $c \in \mathbb{Z}_h$. For each edge i in F'_c, define $y(i) = c$. We need to show that x_1, x_2, y are 3-qualitatively independent. For any $a \in \mathbb{Z}_{g_1}$, $b \in \mathbb{Z}_{g_2}$, $c \in \mathbb{Z}_h$, in the spanning subgraph F'_c there is an edge i incident to $P_a \in P$ and $Q_b \in Q$ as F'_c is a complete bipartite multigraph. This means that for any $a \in \mathbb{Z}_{g_1}$, $b \in \mathbb{Z}_{g_2}$, $c \in \mathbb{Z}_h$, there exists an $i \in [1, n]$ such that $x_1(i) = a$, $x_2(i) = b$, and $y(i) = c$. So, x_1, x_2 and y are 3-qualitatively independent. Next, we prove that x_1 and y are pairwise balanced, and x_2 and y are pairwise balanced. Since F_c is a matching, there is atmost one F_c-edge incident with any of the $\lceil \frac{d_G(P_a)}{h} \rceil$ vertices of H corresponds to $P_a \in P$. Hence

$$d_{F'_c}(P_a) \leq \lceil \frac{d_G(P_a)}{h} \rceil.$$

On the other hand, there are $\lfloor\frac{d_G(P_a)}{h}\rfloor$ points of H corresponds to P_a which have degree h. There must be an F_c-edge starting from each of these, whence

$$d_{F_c'}(P_a) \geq \lfloor\frac{d_G(P_a)}{h}\rfloor.$$

Thus we have $\lfloor\frac{n}{g_1 h}\rfloor \leq d_{F_c{}'}(P_a) \leq \lceil\frac{n}{g_1 h}\rceil$ for $c = 0, 1, \ldots, h-1$. This means that there exist $\lfloor\frac{n}{g_1 h}\rfloor$ or $\lceil\frac{n}{g_1 h}\rceil$ edges $i \in [1, n]$ such that $x_1(i) = a$ and $y(i) = c$, or in other words, each pair of symbols $(a, c) \in \mathbb{Z}_{g_1} \times \mathbb{Z}_h$ between x_1 and y appears either $\lfloor\frac{n}{g_1 h}\rfloor$ or $\lceil\frac{n}{g_1 h}\rceil$ times. So, x_1 and y are pairwise balanced vectors. Similarly, we can show that y and x_2 are pairwise balanced vectors. Next, we need to show that y is balanced. This corresponds to each spanning subgraph F_c' contains either $\lfloor\frac{n}{h}\rfloor$ or $\lceil\frac{n}{h}\rceil$ edges. In other words, this corresponds to each matching F_c contains either $\lfloor\frac{n}{h}\rfloor$ or $\lceil\frac{n}{h}\rceil$ edges. The proof of balancedness is the same as that of Lemma 1.

□

4 Optimal Mixed Covering Array on 3-Uniform Hypergraph

Let H be a vertex-weighted 3-uniform hypergraph with k vertices. Label the vertices $v_1, v_2, ..., v_k$ and for each vertex v_i denote its associated weight by $w_H(v_i)$. Let the *product weight* of H, denoted $PW(H)$, be

$$PW(H) = \max\{w_H(u)w_H(v)w_H(w) : \{u, v, w\} \in E(H)\}.$$

Note that 3-$CAN(H, \prod_{i=1}^{k} w_H(v_i)) \geq PW(H)$. A balanced covering array on H is a covering array on H in which every row is balanced and the rows correspond to vertices in a hyperedge are pairwise balanced.

4.1 Basic Hypergraph Operations

We now introduce four hypergraph operations:

1. *Single-vertex edge hooking I*

2. *Single-vertex edge hooking II*

3. *Two-vertex hyperedge hooking*

4. *Single-vertex hyperedge hooking I*

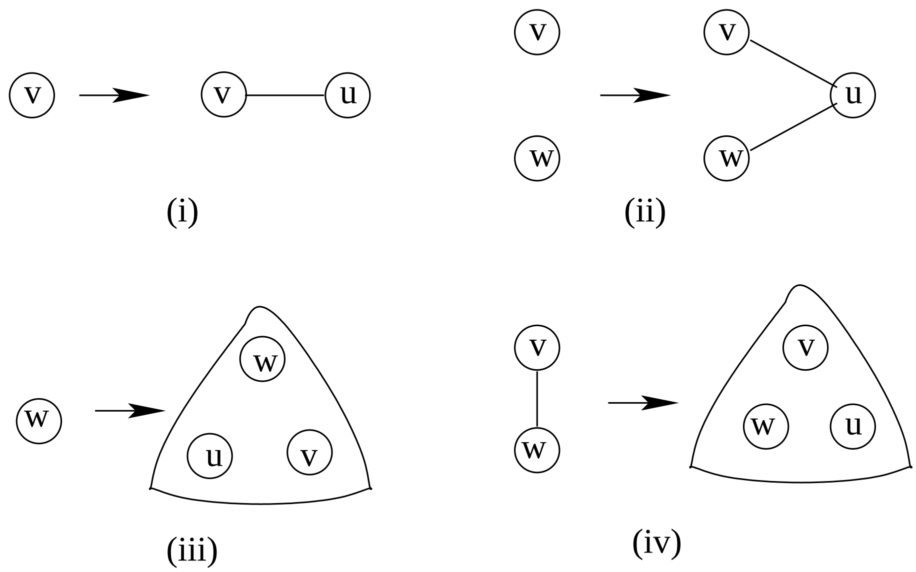

Figure 1: (i) Single-vertex edge hooking I (ii) Single-vertex edge hooking II (iii) Two-vertex hyperedge hooking (iv) Single-vertex hyperedge hooking I

A *single-vertex edge hooking I* in hypergraph H is the operation that inserts a new edge $\{u, v\}$ in which u is a new vertex and v is in $V(H)$. A *single-vertex edge hooking II* in hypergraph H is the operation that inserts two new edges $\{u, v\}$ and $\{u, w\}$ in which u is a new vertex and v and w are in $V(H)$. A *two-vertex hyperedge hooking* in a hypergraph H is the operation that insert a new hyperedge $\{u, v, w\}$ in which u and v are new vertices and w is in $V(H)$. A *single vertex hyperedge hooking I* in a hypergraph H is the operation that replaces an edge $\{v, w\}$ by a hyperedge $\{u, v, w\}$ where u is a new vertex.

Proposition 1. *Let H be a weighted hypergraph with k vertices and H' be the weighted hypergraph obtained from H by single-vertex edge hooking I, single-vertex edge hooking II or single vertex hyperedge hooking I operation with u as a new vertex with $w(u)$ such that $PW(H) = PW(H')$. Then, there exists a balanced $CA(n, H, \prod_{i=1}^{k} g_i)$ if and only if there exists a balanced $CA(n, H', w(u) \prod_{i=1}^{k} g_i)$.*

Proof. If there exists a balanced CA($n, H', w(u) \prod_{i=1}^{k} g_i$) then by deleting the row corresponding to the new vertex u we can obtain a $CA(n, H, \prod_{i=1}^{k} g_i)$. Conversely, let C^H be a balanced CA($n, H, \prod_{i=1}^{k} g_i$). The balanced covering array C^H can be used to construct $C^{H'}$, a balanced CA($n, H', w(u) \prod_{i=1}^{k} g_i$). We consider the following cases:

Case 1: Let H' be obtained from H by a single vertex edge hooking I of a new vertex u with a new edge

$\{u, v\}$, and $w(u)$ such that $w(u)w(v) \leq n$. Using Corollary 1, we can build a balanced length-n vector y corresponds to vertex u such that y is pairwise balanced with the length-n vector x corresponds to vertex v. The array $C^{H'}$ is built by appending row y to C^H.

Case 2: Let H' be obtained from H by a single vertex edge hooking II of a new vertex u with two new edges $\{u, v\}$ and $\{u, w\}$, and $w(u)$ such that $w(u)w(v) \leq n$ and $w(u)w(w) \leq n$. Using Lemma 1, we can build a balanced length-n vector y corresponds to vertex u such that y is pairwise balanced with the length-n vectors x_1 and x_2 correspond to vertices u and v respectively. The array $C^{H'}$ is built by appending row y to C^H.

Case 3: If H' is obtained from H by replacing an edge $\{v, w\} \in E(H)$ by a new hyperedge $\{u, v, w\}$ in which u is a new vertex, and $w(u)$ such that $w(u)w(v)w(w) \leq n$. Using Lemma 2, we can build a balanced length n vector y corresponds to vertex u such that y is 3-qualitatively independent with two length-n pairwise balanced vectors x_1 and x_2 correspond to vertices v and w in H. The array $C^{H'}$ is built by appending row y to C^H. $\square$

Proposition 2. *Let H be a weighted hypergraph with k vertices and H' be the weighted hypergraph obtained from H by two-vertex hyperedge hooking operation with u and v as new vertices with $w(u)$ and $w(v)$ such that $PW(H) = PW(H')$. Then, there exists a balanced $CA(n, H, \prod_{i=1}^{k} g_i)$ if and only if there exists a balanced $CA(n, H^{'}, w(u)w(v) \prod_{i=1}^{k} g_i)$.*

Proof. If there exists a balanced CA($n, H', w(u) \prod_{i=1}^{k} g_i$) then by deleting the rows corresponding to the new vertices u and v we can obtain a $CA(n, H, \prod_{i=1}^{k} g_i)$. Conversely, let C^H be a balanced CA($n, H, \prod_{i=1}^{k} g_i$). Hypergraph H' is obtained from H by a two-vertex hyperedge hooking of two new vertices u and v with a new hyperedge $\{u, v, w\}$, and $w(u)$, $w(v)$ such that $w(u)w(v)w(w) \leq n$. Using Corollary 1, we can build a balanced length-n vector y_1 corresponds to vertex u such that y_1 is pairwise balanced with the length-n vector x corresponds to vertex w. Then using Lemma 2, we can build a balanced length n vector y_2 corresponds to vertex v such that y_2 is 3-qualitatively independent with two length-n pairwise balanced vectors x and y_1 correspond to vertices w and u respectively in H. The array $C^{H'}$ is built by appending rows y_1 and y_2 to C^H.

$\square$

Theorem 2. *Let H be a weighted hypergraph and $H^{'}$ be a weighted 3-uniform hypergraph obtained from H via a sequence of single-vertex edge hooking I, single-vertex edge hooking II, two-vertex hyperedge hooking,*

single-vertex hyperedge hooking I operations. Let $v_{k+1}, v_{k+2}, ..., v_l$ be the vertices in $V(H^{'}) \setminus V(H)$ with weights $g_{k+1}, g_{k+2}, ..., g_l$ respectively so that $PW(H) = PW(H')$. If there exists a balanced covering array $CA(n, H, \prod_{i=1}^{k} g_i)$, then there exists a balanced $CA(n, H^{'}, \prod_{i=1}^{l} g_i)$.

Proof. The result is derived by iterating the different cases of Proposition 1 and Proposition 2. ☐

4.2 α-acyclic 3-uniform hypergraphs

The notion of hypergraph acyclicity plays crucial role in numerous fields of application of hypergraph theory specially in relational database theory and constraint programming. There are many generalizations of the notion of graph acyclicity in hypergraphs. Graham [13], and independently, Yu and Ozsoyoglu [27], defined α-acyclic property for hypergraphs via a transformation now known as the *GYO reduction*. Given a hypergraph $H = (V, E)$, the GYO reduction applies the following operations repeatedly to H until none can be applied:

1. If a vertex $v \in V$ has degree one, then delete v from the edge containing it.
2. If $A, B \in E(H)$ are distinct hyperedges such that $A \subseteq B$, then delete A from $E(H)$.
3. If $A \in E$ is empty, that is $A = \phi$, then delete A from E.

Definition 11. *A hypergraph H is α-acyclic if GYO reduction on H results in an empty hypergraph.*

Example 1. *Hypergraph $H_1 = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 6\}, \{2, 3, 5\}\}$ is α-acyclic.*

Example 2. *Hypergraph $H_2 = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2, 3\}, \{1, 3, 4\}, \{2, 4, 5\}, \{4, 5, 6\}\}$ is not α-acyclic.*

Theorem 3. *Let H be a weighted α-acyclic 3-uniform hypergraph with l vertices. Then there exists a balanced mixed 3-$CA(n, H, \prod_{i=1}^{l} g_i)$ with $n = PW(H)$.*

Proof. Apply the GYO reduction on H to record the order in which the hyperedges are deleted. Let $e_1, e_2, \ldots, e_m$ be a deletion order for the m hyperedges of H. While constructing covering array on H, consider the hyperedges in reverse order of their deletions. Let H_1 be the hypergraph with the single hyperedge

$e_m = \{v_1, v_2, v_3\}$. If $g_1 g_2 g_3 = n$, there exists a trivial balanced covering array $CA(n, H_1, \prod_{i=1}^{3} g_i)$. Otherwise, if $g_1 g_2 g_3 \leq n$, we construct a balanced covering array of size n on H_1 as follows: begin with a balanced vector $x_1 \in \mathbb{Z}_{g_1}^n$ corresponds to vertex v_1. From Proposition 2 (using two-vertex hyperedge hooking operation), we get a balanced covering array $CA(n, H_1, \prod_{i=1}^{3} g_i)$. Let H_2 be the hypergraph obtained from H_1 by adding hyperedge e_{m-1}. Using single-vertex hyperedge hooking I or two-vertex hyperedge hooking operation, there exists a covering array of size n on H_2. For $i = 2, 3, \ldots, m$, let $H_i = H_{i-1} \cup e_{m+1-i}$. Note that $H_m = H$. As $PW(H_i) \leq PW(H)$ for all $i = 2, 3, \ldots, m$, using single-vertex hyperedge hooking I or two-vertex hyperedge hooking operation, there exists a balanced covering array on H_i of size n. In particular, there exists a balanced 3-$CA(n, H, \prod_{i=1}^{l} g_i)$. ☐

Definition 12. *[26] A hypergraph $H = (V, E)$ is called an interval hypergraph if there exists a linear ordering of the vertices $v_1, v_2, ..., v_n$ such that every hyperedge of H induces an interval in this ordering. In other words, the vertices in V can be placed on the real line such that every hyperedge is an interval.*

Corollary 2. *Let H be a weighted 3-uniform interval hypergraph with l vertices. Then there exists a balanced mixed 3-$CA(n, H, \prod_{i=1}^{l} g_i)$ where $n = PW(H)$.*

Proof. This corollary follows immediately from the proof of Theorem 3 since every interval hypergraph is α-acyclic. ☐

4.2.1 3-Uniform Hypertrees

In this subsection, we give a construction for optimal mixed covering arrays on some specific conformal 3-uniform hypertrees. A *host graph* for a hypergraph is a connected graph on the same vertex set, such that every hyperedge induces a connected subgraph of the host graph [26].

Definition 13. (Voloshin [26]). *A hypergraph $H = (V, E)$ is called a $hypertree$ if there exists a host tree $T = (V, E^{'})$ such that each hyperedge $E_i \in E$ induces a subtree of T.*

In other words, any hypertree is isomorphic to some family of subtrees of a tree. A 3-uniform hypertree is a hypertree such that each hyperedge in it contains exactly three vertices.

Theorem 4. *Let H be a weighted conformal 3-uniform hypertree with l vertices, having a binary tree as a host tree. Then there exists a balanced mixed 3-$CA(n, H, \prod_{i=1}^{l} g_i)$ with $n = PW(H)$.*

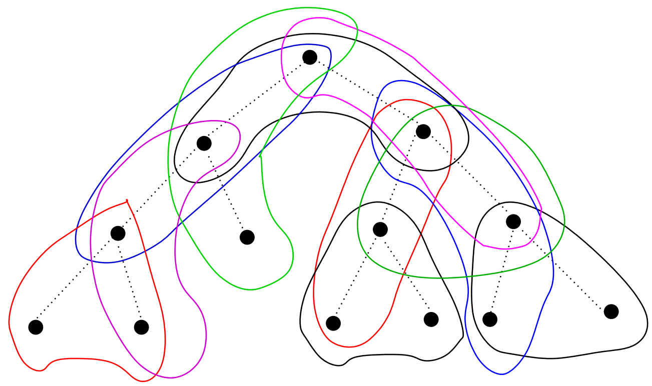

Figure 2: A conformal 3-uniform hypertree with a binary host tree

Proof. We claim that H is an α-acyclic hypergraph. The reason is this. We do not have three hyperedges in H with a common vertex and other 3 vertices pair-wise adjacent as conformality implies a hyperedge of size 4. Thus, H has at least one vertex of degree 1. Apply the GYO reduction on H. At each iteration of the GYO reduction, it produces a partial hypertree which is again a conformal 3-uniform hypertree having a binary tree as host tree. The GYO reduction on H results in an empty hypertree. Therefore, H is an α-acyclic hypergraph. Now the proof follows directly from the proof of Theorem 3. □

4.3 3-uniform Cycles

The cyclic structure is very rich in hypergraphs as compare to that in graphs [1]. It seems difficult to construct optimal size mixed covering arrays on cycle hypergraphs. There are few special types of 3-uniform cycles for which we construct optimal size mixed covering arrays.

Theorem 5. *Let H be a weighted 3-uniform cycle $(v_1, E_1, v_2, E_2, ..., v_k, E_k, v_1)$ of length $k \geq 3$ on $2k$ vertices satisfying the following conditions.*

1. $E_i \cap E_{i+1} = \{v_{i+1}\}$ *for* $i = 1, .., k-1$ *and* $E_k \cap E_1 = \{v_1\}$
2. $d(u_i) = 1$ *for* $u_i \in E_i \setminus \{v_i, v_{i+1}\}$ *where* $i = 1, ..., k-1$ *and* $d(u_k) = 1$ *for* $u_k \in E_k \setminus \{v_k, v_1\}$

Let g_i and ω_i denote the weights of vertices v_i and u_i respectively. Then there exists a balanced 3-$CA(n, H, \prod_{j=1}^{k} g_j\omega_j)$ with $n = PW(H)$.

Proof. Let $\{v_1, u_1, v_2\}$ be a hyperedge in H with $g_1\omega_1 g_2 = PW(H)$. Let H_1 be the hypergraph with the single hyperedge $\{v_1, u_1, v_2\}$. There exists a balanced covering array 3-$CA(n, H_1, \omega_1 \prod_{j=1}^{2} g_j)$. For

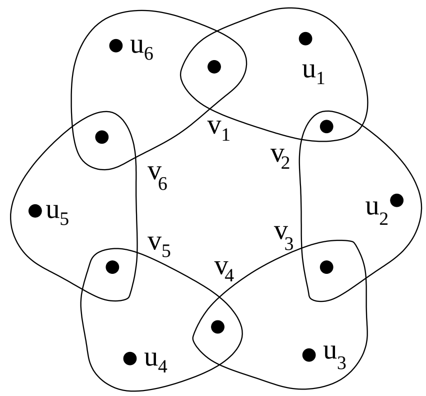

Figure 3: 3-uniform cycle of length-6

$i = 2, 3, \ldots, k-1$, let H_i be the hypergraph obtained from H_{i-1} after inserting a new edge $\{v_i, v_{i+1}\}$ in which v_{i+1} is a new vertex, that is, $H_i = H_{i-1} \cup \{v_i, v_{i+1}\}$. Using Proposition 1 (single-vertex edge hooking I operation), for all $i = 2, 3, \ldots, k-1$, as $g_i g_{i+1} \le n$, there exists a balanced $CA(n, H_i, \omega_1 \prod_{j=1}^{i+1} g_j)$. Let $H_k = H_{k-1} \cup \{\{v_{k-1}, v_k\}, \{v_k, v_1\}\}$. Using single vertex edge hooking II operation, as $g_{k-1} g_k \le n$ and $g_1 g_k \le n$, we get a balanced covering array $CA(n, H_k, \omega_1 \prod_{j=1}^{k} g_j)$. Finally, using sequence of single-vertex hyperedge hooking I operations on H_k, replace edge $\{v_i, v_{i+1}\}$ by hyperedge $\{v_i, u_i, v_{i+1}\}$ for $i = 2, 3, \ldots, k-1$; also replace edge $\{v_k, v_1\}$ by hyperedge $\{v_k, u_k, v_1\}$. As $g_i \omega_i g_{i+1} \le n$ for all $i = 2, 3, \ldots, k-2$ and $g_k \omega_k g_1 \le n$, from Proposition 1 (using single-vertex hyperedge hooking I), there exists a balanced 3-$CA(n, H, \prod_{j=1}^{k} g_j \omega_j)$. $\square$

The length-k 3-uniform cycle considered in Theorem 5 contains k vertices of degree 1. As every hyperedge has one vertex of degree 1, such hypergraph satisfies $|E(H)| = |V(H)|/2$.

5 Further Cycle Hypergraphs

In this section, we consider 3-uniform cycles of length k with exactly one vertex of degree 1. This type of 3-uniform hypergraphs have $|E(H)| = |V(H)| - 2$. Construction of optimal size mixed covering arrays on such cycle hypergraphs seems to be difficult.

Let H be a weighted 3-uniform cycle $(v_0, E_1, v_2, E_2, v_3, E_3, v_0)$ of length-3 on five vertices with $E_1 = \{v_0, v_1, v_2\}$, $E_2 = \{v_1, v_2, v_3\}$ and $E_3 = \{v_3, v_4, v_0\}$ as shown in Figure 5. Let E_1 be a hyperedge in H with $g_0 g_1 g_2 = PW(H)$ where g_i denotes the weight of vertex v_i. Let H_1 be the hypergraph with the single hyperedge E_1. There exists a balanced covering array $CA(n, H_1, \prod_{i=0}^{2} g_i)$ where $n = PW(H)$. Let

$H_2 = H_1 \cup \{E_2\}$. Using Proposition 1 (single-vertex hyperedge hooking I), there exists a balanced covering array $CA(n, H_2, \prod_{i=0}^{3} g_i)$. Let $H_3 = H_2 \cup \{E_3\}$. Note that $H_3 = H$. We cannot use any of the known hypergraph operations to construct a balanced covering array of size $PW(H)$ on H_3 as the rows correspond to v_0 and v_3 are not pairwise balanced. Thus we define a new hypergraph operation called *single vertex hyperedge hooking* II operation. A single vertex hyperedge hooking II in a hypergraph H is the operation that inserts a new hyperedge $\{u, v, w\}$ and a new edge $\{u, z\}$ where $\{v, w, z\}$ is an existing hyperedge in H and u is a new vertex.

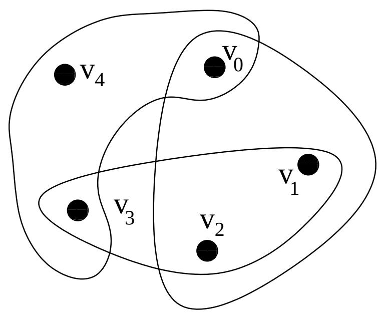

Figure 4: cycle of lenghth 3 with $g_0 = 10, g_1 = 8, g_2 = 5, g_3 = 2, g_4 = 18$

5.1 Balanced Partitioning

Let g_1, g_2, g_3 and $n \geq g_1 g_2 g_3$ be positive integers and $x_1 \in \mathbb{Z}_{g_1}^n$, $x_2 \in \mathbb{Z}_{g_2}^n$ and $x_3 \in \mathbb{Z}_{g_3}^n$ be mutually pairwise balanced and 3-qualitatively independent vectors. Then, we prove in this section, there exists a balanced vector $y \in \mathbb{Z}_h^n$, where h satisfies certain conditions, such that $\{x_1, x_2, y\}$ are 3-qualitatively and y is pairwise balanced with each x_i for $i = 1, 2, 3$.

We construct a tripartite 3-uniform multi-hypergraph G corresponds to x_1, x_2 and x_3 as follows: G has g_1 vertices in the first part $P \subseteq V(G)$, g_2 vertices in the second part $Q \subseteq V(G)$ and g_3 vertices in the third part $R \subseteq V(G)$. Let $P_a = \{i \mid x_1(i) = a\}$ for $a = 0, 1, \ldots, g_1 - 1$, be the vertices of P, $Q_b = \{i \mid x_2(i) = b\}$ for $b = 0, 1, \ldots, g_2 - 1$, be the vertices of Q, and $R_c = \{i \mid x_3(i) = c\}$ for $c = 0, 1, \ldots, g_3 - 1$, be the vertices of R. For each $i = 1, 2, \ldots, n$ there exists exactly one $P_a \in P$ with $i \in P_a$, exactly one $Q_b \in Q$ with $i \in Q_b$ and exactly one $R_c \in R$ with $i \in R_c$. For each such i, add a hyperedge $\{P_a, Q_b, R_c\}$ and label it i. Clearly, $d_G(P_a) = |P_a|$, $d_G(Q_b) = |Q_b|$ and $d_G(R_c) = |R_c|$. Let h

be a positive integer so that $h \leq \min\{\lfloor \frac{n}{g_1 g_2} \rfloor, \lfloor \frac{n}{g_1 g_3} \rfloor\}$ and

$$\lfloor \frac{n}{g_1 g_2} \rfloor \equiv 0 \mod h \quad \text{for } h \geq 3.$$

That is, for each pair $(a, b) \in \mathbb{Z}_{g_1} \times \mathbb{Z}_{g_2}$, the number $d_G(P_a Q_b)$ of hyperedges containing P_a and Q_b is either 0 or 1 mod h. Clearly, $d_G(P_a Q_b) = |P_a \cap Q_b|$. We construct a tripartite 3-uniform hypergraph H with maximum degree h from G as follows: We split each vertex $v \in V(G)$ in G into $\lfloor \frac{d_G(v)}{h} \rfloor$ vertices of degree h and, if necessary, one vertex of degree less than h. Using balancedness of x_1, we have that there are at least g_2 copies of P_a in H from the split operation. Label them $P^l_{a0}, \ldots P^l_{a,g_2-1}, \mathcal{E}^1_a, \mathcal{E}^2_a, \ldots$ for $l = 1, 2, \ldots, \lfloor \frac{d_G(P_a Q_b)}{h} \rfloor$. Similarly, there are at least g_1 copies of Q_b, label them $Q^l_{b0}, \ldots Q^l_{b,g_1-1}, \mathcal{F}^1_b, \mathcal{F}^2_b, \ldots$ for $l = 1, 2 \ldots, \lfloor \frac{d_G(P_a Q_b)}{h} \rfloor$ and at least g_1 copies of R_c; label them $R^l_{c0}, \ldots, R^l_{c,g_1-1}$, $\mathcal{G}^1_c, \mathcal{G}^2_c, \ldots$ for $l = 1, 2 \ldots, \lfloor \frac{d_G(P_a R_c)}{h} \rfloor$.

Each P_a is split as follows: We have either sh or $sh + 1$ hyperedges containing P_a and Q_b for $b = 0, 1, \ldots, g_2 - 1$ where $s = \lfloor \frac{d_G(P_a Q_b)}{h} \rfloor$. Choose a $c \in Z_{g_3}$ (not necessarily distinct for different a). If the number of hyperedges containing P_a and Q_b is $sh + 1$, we pick one hyperedge $i \in P_a \cap Q_b$ so that $x_3(i) = c$. This is possible as x_1, x_2, x_3 are 3-qualitatively independent. Let E_a be the collection of all those hyperedges for $b = 0, 1, \ldots, g_2 - 1$; clearly $|E_a| \leq g_2$. Split E_a into $\lfloor \frac{|E_a|}{h} \rfloor$ vertices of degree h and, if necessary, one vertex of degree less than h. Denote these vertices as $\mathcal{E}^l_a$ for $l = 1, 2, \ldots, \lfloor \frac{|E_a|}{h} \rfloor + 1$. Beside the hyperedges in E_a, we have exactly sh hyperedges containing P_a and Q_b . These sh hyperedges are partitioned into s equal parts. The h hyperedges in one part become h hyperedges containing P^l_{ab} and Q^l_{ba}, $l = 1, 2, \ldots, s$, in H.

Each Q_b is split as follows: For $a = 0, 1, \ldots, g_1 - 1$, set $Q^l_{ba} = P^l_{ab}$. Distribute the remaining elements of Q_b into vertices of degree h and, if necessary, one vertex of degree less than h. Denote these vertices as $\mathcal{F}^l_b$. Each R_c is split as follows: R_c is split so that $\mathcal{E}^l_a \subseteq R^l_{ca}$. Distribute the remaining elements of R_c into vertices of degree h and, if necessary, one vertex of degree less than h. It is easy to observe that this partitioning of P_a, Q_b and R_c is not uniquely determined.

Lemma 3. *H is balanced hypergraph with maximum degree $\Delta(H) = h$.*

Proof. Hypergraph H has $V(H) = X_1 \cup X_2 \cup X_3$ as vertex set where $X_1 = \{P^l_{ab}, \mathcal{E}^l_a \mid a \in \mathbb{Z}_{g_1}, b \in \mathbb{Z}_{g_2}, l \in \mathbb{N}\}$, $X_2 = \{Q^l_{ba}, \mathcal{F}^l_b \mid a \in \mathbb{Z}_{g_1}, b \in \mathbb{Z}_{g_2}, l \in \mathbb{N}\}$ and $X_3 = \{R^l_{ca}, \mathcal{G}^l_c \mid a \in \mathbb{Z}_{g_1}, c \in \mathbb{Z}_{g_3}, l \in \mathbb{N}\}$.

Let $A \subset V(H)$ and H_A be the subhypergraph induced by A. From Theorem 1, it suffices to show that H_A is 2-colourable. Later part of proof deals with 2-colouring of H_A which is based on the following cases.

Case 1: $A \cap X_i = \emptyset$ for two choices of $i \in \{1, 2, 3\}$. Without loss of generality we assume $A \cap X_1 = \emptyset$ and $A \cap X_2 = \emptyset$, that is, A intersects only with X_3. Being H a tripartite hypergraph, A is an independent set in this case and H_A has no hyperedges. Hence it is 2-colourable.

Case 2: $A \cap X_i = \emptyset$ for exactly one i. Without loss of generality we assume $A \cap X_1 = \emptyset$. As A intersects with X_2 and X_3, the induced sub-hypergraph H_A is a bipartite graph between X_2 and X_3. Hence H_A is 2-colourable.

Case 3: $A \cap X_i \neq \emptyset$ for all i. We claim that H_A is union of a 3-uniform partial hypergraph of H and a bipartite graph on A. Every partial hypergraph of H is 2-colourable as H is 2-colourable. Consider a 2-colouring of bipartite graph induced by subhypergraph and extend this to 2-colouring of 3-uniform partial hypergraph to produce a 2-colouring of H_A. To show that subgraph induced by A is a bipartite graph consider a 2-uniform cycle C in H_A. If C does not intersect some X_i then it alternates between vertices of only two partite sets and turns out as a bipartite graph. Now we assume C intersects each partite set X_1, X_2 and X_3. Consider a vertex $v \in C \cap X_1$. We denote by $N_H(v)$ the set of neighbours of v in H. There are two types of vertices in X_1 either of the form P^l_{ab} or of the form E^l_a. If v is P^l_{ab} then $N_H(v) \cap X_2$ has only one vertex which is Q^l_{ba}. Hence the edge $P^l_{ab} Q^l_{ba}$ cannot be part of any cycle in H_A. Consequently both neighbours of P^l_{ab} in C are from X_3 and corresponding incident edges in C are induced only if $Q^l_{ba} \notin A$. If v is $\mathcal{E}^l_a$ then $N_H(v) \cap X_3$ has only one vertex which is some R^l_{ca}. Hence the edge $\mathcal{E}^l_a R^l_{ca}$ cannot be part of cycle C. Consequently both neighbours of $\mathcal{E}^l_a$ in C are from X_2 and corresponding incident edges in C are induced only if $R^l_{ca} \notin A$. Thus either $N_C(v) \subset X_2$ or $N_C(v) \subset X_3$. We identify the neighbours $N_C(v) \in X_i$ as a single vertex $N(v)$ from X_i. This identification operation reduces the length of C by two and creates a smaller cycle with v hanging out side of this new cycle by an edge incident at $N(v)$ with multiplicity 2. After performing identification for each $v \in C \cap X_1$, we left with a cycle C' that alternates between vertices in X_2 and X_3. Consequently C' has to be of even length. Each identification operation reduces the length of C by 2 whence total reduction in length is even. The length of C is equal to sum of length of C' and the total reduction and hence it is an even integer. This shows that H_A does not contain any odd length 2-uniform cycle. □

Definition 14. *[2] A matching in a hypergraph H is a family of pairwise disjoint hyperedges. In other*

words matching is a partial hypergraph H_0 with maximum degree $\Delta(H_0) = 1$.

Theorem 6. *[3] The hyperedges of a balanced hypergraph H with maximum degree Δ, can be partitioned into Δ matchings.*

Lemma 4. *Let $x_1 \in \mathbb{Z}_{g_1}^n$, $x_2 \in \mathbb{Z}_{g_2}^n$ and $x_3 \in \mathbb{Z}_{g_3}^n$ be mutually pairwise balanced and 3-qualitatively independent vectors. Let h be a positive integer so that $h \leq \min\{\lfloor \frac{n}{g_1 g_2} \rfloor, \lfloor \frac{n}{g_1 g_3} \rfloor\}$ and for $h \geq 3$,*

$$\lfloor \frac{n}{g_1 g_2} \rfloor \equiv 0 \mod h.$$

Then there exists a balanced vector $y \in \mathbb{Z}_h^n$ such that $\{x_1, x_2, y\}$ are 3-qualitatively independent and y is pairwise balanced with each x_i for $i = 1, 2, 3$.

Proof. Construct a tripartite 3-uniform hypergraph H corresponding to x_1, x_2 and x_3 as described above. Lemma 3 implies that H is a balanced hypergraph having maximum degree $\Delta(H) = h$. Theorem 6 says that $E(H)$ is union of h edge-disjoint matching $F_0, F_1, \ldots, F_{h-1}$. Identify those points of H which corresponds to the same point of G, then F_0, F_1, $\ldots$, F_{h-1} are mapped onto certain edge disjoint spanning partial hypergraphs F_0', F_1', $\ldots$, F_{h-1}' of G. These h edge-disjoint spanning partial hypergraphs F_0', F_1', $\ldots$, F_{h-1}' of G form a partition of $E(G) = [1, n]$ which we use to build a balanced vector $y \in \mathbb{Z}_h^n$. Each edge disjoint spanning partial hypergraph corresponds to a symbol in $\mathbb{Z}_h$ and each edge corresponds to an index from $[1, n]$. Suppose edge disjoint spanning partial hypergraph F_d' corresponds to symbol $d \in \mathbb{Z}_h$. For each edge i in F_d', define $y(i) = d$. We have

$$\lfloor \frac{n}{g_1 h} \rfloor \leq d_{F_d{}'}(P_a) \leq \lceil \frac{n}{g_1 h} \rceil$$

for $d = 0, 1, \ldots, h-1$. It follows from similar arguments as in Lemma 2. Similarly y is pairwise balanced with x_2 and x_3. Now we show that x_1, x_2, y are 3-qualitatively independent. Let $(a, b, d) \in \mathbb{Z}_{g_1} \times \mathbb{Z}_{g_2} \times \mathbb{Z}_h$ be a tuple of symbols. For every $a \in \mathbb{Z}_{g_1}$, $b \in \mathbb{Z}_{g_2}$, there are h hyperedges containing P_{ab}^l and Q_{ba}^l in H, and they will all appear in different matchings F_0, F_1, $\ldots$, F_{h-1}. This ensures that each spanning partial hypergraph contains at least one $P_a - Q_b$ hyperedge for every $a \in \mathbb{Z}_{g_1}, b \in \mathbb{Z}_{g_2}$. Whence there exists at least one hyperedge $i \in F_d'$ such that $x_1(i) = a$, $x_2(i) = b$ and $y(i) = d$. Thus, x_1, x_2 and y are 3-qualitatively independent. We need to show that y is balanced. This corresponds to each matching F_i contains either $\lfloor \frac{n}{h} \rfloor$ or $\lceil \frac{n}{h} \rceil$ hyperedges. Suppose we have two matching F_0 and F_1 that differ by size more than 1, say F_0 smaller and F_1 larger. Every component of the union of F_0 and F_1 could be an alternating even cycle hypergraph or

alternating path. Note that it must contain a path, otherwise their sizes are equal. We can find an alternating path in the union hypergraph that contains more edges from F_1 than F_0. Swap the F_1 edges with the F_0 edges in this alternating path. Then the resultant graph has F_0 increased in size by 1 hyperedge, and F_1 decreased in size by 1 hyperedge. Continue this process on F_0, F_1, ..., F_{h-1} until the sizes are correct. $\square$

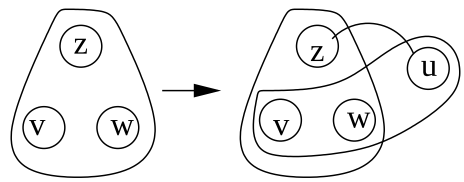

Figure 5: Single-vertex hyperedge hooking II

Proposition 3. *Let H be a weighted hypergraph with k vertices and H' be the weighted hypergraph obtained from H by single vertex hyperedge hooking II operation with u as a new vertex with $w(u)$ such that $PW(H) = PW(H')$ and for $w(u) \geq 3$*

$$\lfloor \frac{n}{w(v)w(w)} \rfloor \equiv 0 \quad mod \;\; w(u).$$

Then, there exists a balanced $CA(n, H, \prod_{i=1}^{k} g_i)$ if and only if there exists a balanced $CA(n, H^{'}, w(u)\prod_{i=1}^{k} g_i)$.

Proof. If there exists a balanced CA($n, H', w(u)\prod_{i=1}^{k} g_i$) then by deleting the row corresponding to the new vertex u we can obtain a $CA(n, H, \prod_{i=1}^{k} g_i)$. Conversely, let C^H be a balanced CA($n, H, \prod_{i=1}^{k} g_i$). If H' is obtained from H by a single vertex hyperedge hooking II of a new vertex u with a new hyperedge $\{u, v, w\}$ and a new edge $\{u, z\}$ where $\{v, w, z\}$ is an existing hyperedge in H and $w(u)$ such that $w(u)w(v)w(w) \leq n$ and $w(u)w(z) \leq n$. Using Lemma 4, we can build a length-n vector y such that $\{y, x_1, x_2\}$ is 3-qualitatively independent and y is pairwise balanced with x_1, x_2, x_3, where x_1, x_2, x_3 are length-n vectors correspond to vertices v, w, z respectively. The array $C^{H'}$ is obtained by appending row y to C^H. $\square$

Theorem 7. *Let H be a weighted 3-uniform cycle $(v_0, E_1, v_2, E_2, v_3, E_3, v_0)$ of length-3 on five vertices with $E_1 = \{v_0, v_1, v_2\}$, $E_2 = \{v_1, v_2, v_3\}$ and $E_3 = \{v_3, v_4, v_0\}$. Let g_i denote the weight of vertex v_i. Let E_1 be a hyperedge in H with $g_0g_1g_2 = PW(H)$. If $g_0 \equiv 0 \mod g_3$ and $g_3 \leq \min\{g_0, \max\{g_1, g_2\}\}$ then there exists a balanced 3-$CA(n, H, \prod_{i=0}^{4} g_i)$ with $n = PW(H)$.*

Proof. Let H_1 be a hypergraph with single hyperedge E_1. There exists a balanced 3-$CA(n, H_1, \prod_{i=0}^{2} g_i)$. Let $H_2 = H_1 \cup \{E_2, \{v_0, v_3\}\}$. From Proposition 3, as $g_0 \equiv 0 \mod g_3$ and $g_3 \leq \min\{g_0, \max\{g_1, g_2\}\}$,

there exists a balanced 3-$CA(n, H_2, \prod_{i=0}^{3} g_i)$. Let H_3 be the hypergraph obtained from H_2 by replacing edge $\{v_0, v_3\}$ by hyperedge $\{v_0, v_3, v_4\}$. Note that $H_2 = H$. As $g_0 g_3 g_4 \leq n$, using single-vertex hyperedge hooking I operation, we get a balanced covering array 3-$CA(n, H, \prod_{i=0}^{4} g_i)$ $\square$

6 Conclusions and Open Problems

In this paper, we study construction of optimal mixed covering arrays on 3-uniform hypergrahs. This paper extends the work done by Meagher, Moura, and Zekaoui [20] for mixed covering arrays on graph to mixed covering arrays on hypergarphs. We gave five hypergraph operations that enable us to add new vertices, edges and hyperedges to a hypergraph. These operations have no effect on the covering array number of the modified hypergraph. Using these hypergraph operations, we build optimal mixed covering arrays for special classes of hypergraphs, e.g., 3-uniform α-acyclic hypergraphs, 3-uniform interval hypergraphs, 3-uniform conformal hypertrees, and specific 3-uniform cycles. The five basic hypergraph operations introduced here may be useful for obtaining optimal mixed covering arrays on other classes of hypergraphs. It is an interesting open problem to find optimal mixed covering arrays on conformal hypergraphs, tight cycle hypergraphs, Steiner triple systems, etc.

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